UNIVERSAL C^*-ALGEBRA OF REAL RANK ZERO

ALEX CHIGOGIDZE

ABSTRACT. It is well-known that every commutative separable unital C^*-algebra of real rank zero is a quotient of the C^*-algebra of all compex continous functions defined on the Cantor cube. We prove a non-commutative version of this result by showing that the class of all separable unital C^*-algebras of real rank zero concides with the class of quotients of a certain separable unital C^*-algebra of real-rank zero.

1. INTRODUCTION

It is a well-known observation (see, for instance, [5, Theorem 1.3.15]) that the Cantor cube $\{0,1\}^\omega$ contains topological copy of any zero-dimensional metrizable compact space. By Gelfand's duality and by [2, Proposition 1.1], this means that every commutative separable unital C^*-algebra of real rank zero is a quotient of the C^*-algebra $C(\{0,1\}^\omega)$. Our goal in the present note is to extend this result to the non-commutative case.

For a given class $\mathcal{C}$ of separable unital C^*-algebras an element $Z \in \mathcal{C}$ is said to be universal (in $\mathcal{C}$) if every other element $X \in \mathcal{C}$ can be represented as the image of Z under a surjective unital $*$-homomorphism. One particular method of obtaining universal elements for the class $\mathcal{C}$ is to prove the existence of $\mathcal{C}$-invertible morphism $p\colon C^*(\mathbb{F}_\infty) \to Z$. Here $C^*(\mathbb{F}_\infty)$ stands for the group C^*-algebra of the free group on countable number of generators and the $\mathcal{C}$-invertibility of p means that for any unital $*$-homomorphism $g\colon C^*(\mathbb{F}_\infty) \to X$, with $X \in \mathcal{C}$, there exists a unital $*$-homomorphism $h\colon Z \to X$ such that $g = h \circ p$. It is easy to see that in such a situation Z is an universal element in the class $\mathcal{C}$. Indeed, every element X of $\mathcal{C}$ (and, in general, every separable unital C^*-algebra) can be represented as the image of $C^*(\mathbb{F}_\infty)$ under a unital $*$-homomorphism $g\colon C^*(\mathbb{F}_\infty) \to X$. By the $\mathcal{C}$-invertibility of p and by the fact that $X \in \mathcal{C}$, there exists a unital $*$-homomorphism $h\colon Z \to X$ such that $g = h \circ p$. Since g is surjective it follows from the latter equality that h is surjective as well.

In the present note we consider the case $\mathcal{C} = \mathcal{RR}_0$, where $\mathcal{RR}_0$ denotes the class of all separable unital C^*-algebras of real rank zero, and prove that there

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indeed exists a $\mathcal{RR}_0$-invertible morphism $p\colon C^*(\mathbb{F}_\infty) \to Z$ such that $Z \in \mathcal{RR}_0$ which, as noted, implies that Z is universal C^*-algebra in the class $\mathcal{RR}_0$.

2. Preliminaries

All C^*-algebras below are assumed to be unital and all $*$-homomorphisms between unital C^*-algebras are also assumed to be unital. When we refer to a unital C^*-subalgebra of a unital C^*-algebra we implicitly assume that the inclusion is a unital $*$-homomorphism. The set of all self-adjoint elements of a C^*-algebra X is denoted by X_{sa}. The density $d(X)$ of a C^*-algebra X is the minimal cardinality of dense subspaces (in a purely topological sense) of X. Thus $d(X) \leq \omega$ (ω denotes the first infinite cardinal number) means that X is separable.

2.1. **Set-theoretical facts.** For the reader's convenience we begin by presenting necessary set-theoretic facts. Their complete proofs can be found in [4].

Let A be a partially ordered *directed set* (i.e. for every two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A *majorates* another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain C of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A.

The standard example of a τ-complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A. There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets of the set A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ-complete set.

Proposition 2.1. *Let $\{A_t : t \in T\}$ be a collection of τ-closed and cofinal subsets of a τ-complete set A. If $\mid T \mid \leq \tau$, then the intersection $\cap\{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ-closed in A .*

Corollary 2.2. *For each subset B, with $\mid B \mid \leq \tau$, of a τ-complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

Proposition 2.3. *Let A be a τ-complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*
Majorantness: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*

τ-closeness: *Let $\{\alpha_t : t \in T\}$ be a chain in A with $|T| \leq \tau$. If $(\alpha_t, \beta) \in L$ for some $\beta \in A$ and each $t \in T$, then $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t \colon t \in T\}$.*

Then the set $\{\alpha \in A \colon (\alpha, \alpha) \in L\}$ is cofinal and τ-closed in A.

2.2. **Direct C^*_τ-systems of C^*-algebras.** Recall (see [8, Section 1.23] for details) that a direct system $\mathcal{S} = \{X_\alpha, i^\beta_\alpha, A\}$ of unital C^*-algebras consists of a partially ordered directed indexing set A, unital C^*-algebras X_α, $\alpha \in A$, and unital $*$-homomorphisms $i^\beta_\alpha \colon X_\alpha \to X_\beta$, defined for each pair of indexes $\alpha, \beta \in A$ with $\alpha \leq \beta$, and satisfying the condition $i^\gamma_\alpha = i^\gamma_\beta \circ i^\beta_\alpha$ for each triple of indexes $\alpha, \beta, \gamma \in A$ with $\alpha \leq \beta \leq \gamma$. The (inductive) limit of the above direct system is a unital C^*-algebra which is denoted by $\varinjlim \mathcal{S}$. For each $\alpha \in A$ there exists a unital $*$-homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ which will be called the α-th limit homomorphism of $\mathcal{S}$.

If A' is a directed subset of the indexing set A, then the subsystem $\{X_\alpha, i^\beta_\alpha, A'\}$ of $\mathcal{S}$ is denoted $\mathcal{S}|A'$.

Below in in Section 3 we use the concept of the direct C^*_τ-system introduced in [3].

Definition 2.4. Let $\tau \geq \omega$ be a cardinal number. A direct system $\mathcal{S} = \{X_\alpha, i^\beta_\alpha, A\}$ of unital C^*-algebras X_α and unital $*$-homomorphisms $i^\beta_\alpha \colon X_\alpha \to X_\beta$ is called a *direct C^*_τ-system* if the following conditions are satisfied:

(a) A is a τ-complete set.
(b) Density of X_α is at most τ (i.e. $d(X_\alpha) \leq \tau$), $\alpha \in A$.
(c) The α-th limit homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ is an injective $*$-homomorphism for each $\alpha \in A$.
(d) If $B = \{\alpha_t \colon t \in T\}$ is a chain of elements of A with $|T| \leq \tau$ and $\alpha = \sup B$, then the limit homomorphism $\varinjlim\{i^\alpha_{\alpha_t} \colon t \in T\} \colon \varinjlim(\mathcal{S}|B) \to X_\alpha$ is an isomorphism.

Proposition 2.5 (Proposition 3.2, [3])**.** *Let τ be an infinite cardinal number. Every unital C^*-algebra X can be represented as the limit of a direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ where the indexing set A coincides with $\exp_\tau Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.*

Proof. If $d(X) \leq \tau$, then consider the direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \exp_\tau d(X)\}$, where $X_\alpha = X$ for each $\alpha \in \exp_\tau d(X)$ and $i^\beta_\alpha = \mathrm{id}_X$ for each $\alpha, \beta \in \exp_\tau d(X)$ with $\alpha \leq \beta$.

If $d(X) > \tau$, then consider any subset Y of X such that $\mathrm{cl}_X Y = X$ and $|Y| = d(X)$. Without loss of generality we may assume that Y contains the unit of X. Each $\alpha \in \exp_\tau d(X)$ can obviously be identified with a subset (denoted by the same letter α) of Y of cardinality $\leq \tau$. Let X_α be the smallest C^*-subalgebra of X containing α. If $\alpha, \beta \in \exp_\tau d(X)$ and $\alpha \leq \beta$, then $\alpha \subseteq \beta$ (as subsets of Y) and consequently $X_\alpha \subseteq X_\beta$. This inclusion map is denoted by

$i^\beta_\alpha \colon X_\alpha \to X_\beta$. It is easy to verify that the collection $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \exp_\tau d(X)\}$ is indeed a direct C^*_τ-system such that $\varinjlim \mathcal{S}_X = X$. $\square$

Lemma 2.6 (Lemma 3.3, [3])**.** *If $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ is a direct C^*_τ-system, then*

$$\varinjlim \mathcal{S}_X = \bigcup \{i_\alpha(X_\alpha) \colon \alpha \in A\}.$$

Proof. Clearly $\bigcup \{i_\alpha(X_\alpha) \colon \alpha \in A\}$ is dense in $\varinjlim \mathcal{S}_X$ (this fact remains true for arbitrary direct systems of C^*-algebras). Consequently, for any point $x \in \varinjlim \mathcal{S}_X$ there exists a sequence $\{x_n \colon n \in \omega\}$, consisting of elements from $\bigcup \{i_\alpha(X_\alpha) \colon \alpha \in A\}$, such that $x = \lim\{x_n \colon n \in \omega\}$. For each $n \in \omega$ choose an index $\alpha_n \in A$ such that $x_n \in i_{\alpha_n}(X_{\alpha_n})$. By Corollary 2.2, there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Since $i_{\alpha_n} = i_\alpha \circ i^\alpha_{\alpha_n}$, it follows that

$$x_n \in i_{\alpha_n}(X_{\alpha_n}) = i_\alpha\left(i^\alpha_{\alpha_n}(X_{\alpha_n})\right) \subseteq i_\alpha(X_\alpha) \quad \text{for each} \quad n \in \omega.$$

Finally, since $i_\alpha(X_\alpha)$ is closed in $\varinjlim \mathcal{S}_X$, it follows that

$$x = \lim\{x_n \colon n \in \omega\} \in i_\alpha(X_\alpha).$$

$\square$

3. Existence of an universal separable unital C^*-algebra of real rank zero

The real rank of a unital C^*-algebra A, denoted by $RR(A)$, is defined as follows [2]. We say that $RR(A) \leq n$ if for each $(n+1)$-tuple $(x_1, \dots, x_{n+1})$ of self-adjoint elements in A and every $\epsilon > 0$, there exists an $(n+1)$-tuple $(y_1, \dots, y_{n+1})$ in A_{sa} such that $\sum y_k^2$ is invertible and

$$\left\| \sum_{k=1}^{n+1} (x_k - y_k)^2 \right\| < \epsilon.$$

Obviously unital C^*-algebras of real rank zero are defined as those in which every self-adjoint element can be arbitrarily closely approximated by self-adjoint invertible elements.

Proposition 3.1. *Let $\tau \geq \omega$ and X be an unital C^*-algebra. Then the following conditions are equivalent:*

1. *$RR(X) = 0$.*
2. *X can be represented as the direct limit of a direct C^*_τ-system $\{X_\alpha, i^\beta_\alpha, A\}$ satisfying the following properties:*
   (a) *The indexing set A is cofinal and τ-closed in the τ-complete set $\exp_\tau Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.*
   (b) *X_α is a C^*-subalgebra of X such that $RR(X_\alpha) = 0$, $\alpha \in A$.*

Proof. The implication (2) $\Longrightarrow$ (1) follows from [2, Proposition 3.1].

In order to prove the implication (1) $\Longrightarrow$ (2) we first consider a direct C^*_τ-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, A\}$ with properties indicated in Proposition 2.5. Next consider the following relation $L \subseteq A^2$:

$$L = \left\{(\alpha, \beta) \in A^2 \colon \alpha \leq \beta \text{ and for each } \epsilon > 0 \text{ and for each } x \in (X_\alpha)_{sa} \text{ there} \right.$$
$$\left. \text{exists an invertible } y \in (X_\beta)_{sa} \text{ such that } \|y - x\| < \epsilon \right\}.$$

Let us verify conditions of Proposition 2.3.

Existence. Let $\alpha \in A$ and $x \in (X_\alpha)_{sa}$. First we prove the following assertion.

$(*)_{\left(\alpha,x,\frac{1}{n}\right)}$ There exist an index $\beta = \beta\left(\alpha, x, \frac{1}{n}\right) \in A$, $\beta \geq A$ and an invertible element $y = y\left(\alpha, x, \frac{1}{n}\right) \in (X_\beta)_{sa}$ such that $\|y - x\| < \dfrac{1}{n}$

Proof of $(*)_{\left(\alpha,x,\frac{1}{n}\right)}$. By (1), $RR(X) = 0$ and consequently there exists an invertible element $y \in X_{sa}$ such that $\|y - x\| < \dfrac{1}{n}$. Since $\mathcal{S}_X$ is a direct C^*_τ-system it follows from Lemma 2.6 that there exists an index $\alpha' \in A$ such that $y \in X_{\alpha'}$. Since A is a directed set there exists an index $\beta = \beta\left(\alpha, x, \frac{1}{n}\right) \in A$ such that $\alpha, \alpha' \leq \beta$. This obviously implies that $X_\beta \supseteq X_\alpha \cup X_{\alpha'} \supseteq X_\alpha \cup \{y\}$. This finishes the proof of $(*)_{\left(\alpha,x,\frac{1}{n}\right)}$.

For a given element $x \in (X_\alpha)_{sa}$ consider indeces $\beta\left(\alpha, x, \frac{1}{n}\right) \in A$, satisfying conditions $(*)_{\left(\alpha,x,\frac{1}{n}\right)}$, $n = 1, 2, \dots$. By Corollary 2.2, there exists an element $\beta(\alpha, x) \in A$ such that $\beta(\alpha, x) \geq \beta\left(\alpha, x, \frac{1}{n}\right)$ for each $n = 1, 2, \dots$. Obviously $X_\alpha \subseteq X_{\beta(\alpha,x)}$, $n = 1, 2, \dots$. Note also that

$(*)_{(\alpha,x)}$ For any $\epsilon > 0$ there exists an invertible element $y \in \left(X_{(\alpha,x)}\right)_{sa}$ such that $\|y - x\| < \epsilon$.

Since $\mathcal{S}_X$ is a direct C^*_τ-system it follows that $d(X_\alpha) \leq \tau$. Consequently there exists a subset $Y_\alpha \subseteq (X_\alpha)_{sa}$ such that $(X_\alpha)_{sa} = \mathrm{cl}_{(X_\alpha)_{sa}} Y_\alpha$ and $|Y_\alpha| = d\left((X_\alpha)_{sa}\right) \leq d(X_\alpha) \leq \tau$. Let $Y_\alpha = \{y_\gamma \colon \gamma < \tau\}$. For each $\gamma < \tau$ consider an index $\beta(\alpha, y_\gamma) \in A$ satisfying condition $(*)_{(\alpha,y_\gamma)}$. Since A is a τ-complete set, we conclude by Corollary 2.2, there exists an index $\beta = \beta(\alpha) \in A$ such that $\beta \geq \beta(\alpha, y_\gamma)$ for each $\gamma \in \tau$.

We claim that $(\alpha, \beta) \in L$. Indeed, let $\epsilon > 0$ and $x \in (X_\alpha)_{sa}$. Since, by the above construction, $Y_\alpha \subseteq (X_\alpha)_{sa}$ is dense in $(X_\alpha)_{sa}$, there exists a self-adjoint element $y_\gamma \in Y_\alpha$ such that $\|y_\gamma - x\| \leq \dfrac{\epsilon}{2}$. By condition $(*)_{(\alpha,y_\gamma)}$ there exists an invertible element $y \in \left(X_{(\alpha,x)}\right)_{sa}$ such that $\|y - y_\gamma\| < \dfrac{\epsilon}{2}$. Since $\beta(\alpha) \geq \beta(\alpha, y_\gamma)$, it follows that $X_{\beta(\alpha,y_\gamma)} \subseteq X_{\beta(\alpha)}$. This guarantees that $y \in \left(X_{\beta(\alpha)}\right)_{sa}$. It only remains to note that $\|x - y\| < \epsilon$. Therefore $(\alpha, \beta) \in L$.

Majorantness. Let $(\alpha, \beta) \in L$, $\gamma \geq \beta$, $\epsilon > 0$ and $x \in (X_\alpha)_{sa}$. Since $(\alpha, \beta) \in L$, there exists an invertible element $y \in (X_\beta)_{sa}$ such that $\|y - x\| < \epsilon$. Since $\gamma \geq \beta$ it follows that $(X_\beta)_{sa} \subseteq (X_\gamma)_{sa}$ which shows that $y \in (X_\gamma)_{sa}$ and proves that $(\alpha, \gamma) \in L$.

τ-closeness. Suppose that $\{\alpha_t : t \in T\}$ is a chain of indeces in A with $|T| \leq \tau$. Assume also that $(\alpha_t, \beta) \in L$ for each $t \in T$ where $\beta \in A$. Our goal is to show that $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t \colon t \in T\}$. Let $\epsilon > 0$ and $x \in X_\alpha$. Since $\mathcal{S}_X$ is a direct C^*_τ-system it follows that X_α is the direct limit of the direct system generated by C^*-subalgebras X_{α_t}, $t \in T$, and corresponding inclusion homomorphisms. Consequently there exist an index $t \in T$ and an element $x_t \in (X_{\alpha_t})_{sa}$ such that $\|x - x_t\| < \dfrac{\epsilon}{2}$. Since $(\alpha_t, \beta) \in L$ there exists an invertible element $y \in (X_\beta)_{sa}$ such that $\|x_t - y\| < \dfrac{\epsilon}{2}$. Clearly $\|x - y\| < \epsilon$. This shows that $(\alpha, \beta) \in L$.

We are now in position to apply Proposition 2.3 which guarantees that the set $A' = \{\alpha \in A \colon (\alpha, \alpha) \in L\}$ is cofinal and τ-closed in A. Note here that $(\alpha, \alpha) \in L$ precisely when for each $\epsilon > 0$ and for each element $x \in (X_\alpha)_{sa}$ there exists an invertible element $y \in (X_\alpha)_{sa}$ such that $\|x - y\| < \epsilon$. This means that the direct C^*_τ-system $\varinjlim \mathcal{S}'_X = \{X_\alpha, i^\beta_\alpha, A'\}$ consists of C^*-subalgebras of X of real rank zero. Clearly $\varinjlim \mathcal{S}'_X = X$. Proof is completed. $\square$

Corollary 3.2. *The following conditions are equivalent for any unital C^*-algebra X:*

1. *$RR(X) = 0$.*
2. *X can be represented as the direct limit of a direct C^*_ω-system $\{X_\alpha, i^\beta_\alpha, A\}$ satisfying the following properties:*
   (a) *The indexing set A is cofinal and ω-closed in the ω-complete set $\exp_\omega Y$ for some (any) dense subset Y of X with $|Y| = d(X)$.*
   (b) *X_α is a separable unital C^*-subalgebra of X such that $RR(X_\alpha) = 0$, $\alpha \in A$.*

Lemma 3.3. *Let $\{X_t \colon t \in T\}$ be an arbitrary collection of C^*-algebras and $RR(X_t) = 0$ for each $t \in T$. Then $RR\left(\prod\{X_t \colon t \in T\}\right) = 0$.*

Proof. This is an elementary exercise. Let $x = \{x_t \colon t \in T\} \in \prod\{X_t \colon t \in T\}$ be a self-adjoint element of the product $\prod\{X_t \colon t \in T\}$. This obviously means that $x_t = x_t^*$ for each $t \in T$. Let also $\epsilon > 0$. Since $RR(X_t) = 0$, $t \in T$, it follows that there exists a self-adjoint and invertible element $y_t \in X_t$ such that $\|x_t - y_t\| < \dfrac{\epsilon}{2}$, $t \in T$. Obviously $y = \{y_t \colon t \in T\} \in \prod\{X_t \colon t \in T\}$ and $\|x - y\| < \epsilon$ as required. $\square$

Next we construct a universal separable unital C^*-algebra Z of real rank zero. Universal in the sense that any other separable unital C^*-algebra with real rank zero is a quotient of Z. We note here that the group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable number of generators is certainly universal but its real rank is greater than zero. To see the latter assume the contrary, i.e. $RR(C^*(\mathbb{F}_\infty)) = 0$. By [6], $\mathrm{cer}(C^*(\mathbb{F}_\infty)) < 1 + \epsilon$ which contradicts the fact [7, p.370] that $\mathrm{cer}(C^*(\mathbb{F}_\infty)) = \infty$ (here cer stands for an exponential rank). Therefore $RR(C^*(\mathbb{F}_\infty)) > 0$.

Theorem 3.4. *The class $\mathcal{RR}_0$ of all separable unital C^*-algebras of real rank zero contains an universal element Z. More formally, there exists a $\mathcal{RR}_0$-invertible unital $*$-homomorphism $p\colon C^*(\mathbb{F}_\infty) \to Z$ where Z is a separable unital C^*-algebra such that $RR(Z) = 0$.*

Proof. Let $\mathcal{A} = \{f_t\colon C^*(\mathbb{F}_\infty) \to X_t, t \in T\}$ denote the set (see [1, Proposition 2.1]) of all unital $*$-homomorphisms defined on $C^*(\mathbb{F}_\infty)$ such that $RR(X_t) = 0$. Next consider the product $\prod\{X_t\colon t \in T\}$. Since $RR(X_t) = 0$ for each $t \in T$ it follows from Lemma 3.3 that $RR\left(\prod\{X_t\colon t \in T\}\right) = 0$. The unital $*$-homomorphisms f_t, $t \in T$, define the unique unital $*$-homomorphism

$$f\colon C^*(\mathbb{F}_\infty) \to \prod\{X_t\colon t \in T\}$$

such that $\pi_t \circ f = f_t$ for each $t \in T$ (here $\pi_t\colon \prod\{X_t\colon t \in T\} \to X_t$ denotes the corresponding canonical projection $*$-homomorphism). By Corollary 3.2, $\prod\{X_t\colon t \in T\}$ can be represented as the limit of the C^*_ω-system $\mathcal{S} = \{C_\alpha, i^\beta_\alpha, A\}$ such that C_α is a separable unital C^*-algebra of real rank zero for each $\alpha \in A$. Suppressing injective unital $*$-homomorphisms $i^\beta_\alpha\colon C_\alpha \to C_\beta$ we for notational simplicity assume that C_α's are unital C^*-subalgebras of $\prod\{X_t\colon t \in T\}$. Let $\{a_n\colon n \in \omega\}$ be a countable dense subset of $C^*(\mathbb{F}_\infty)$. By Lemma 2.6, for each $n \in \omega\}$ there exists an index $\alpha_n \in A$ such that $f(a_n) \in C_{\alpha_n}$. By Corollary 2.2, there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Then $f(a_n) \in C_{\alpha_n} \subseteq C_\alpha$ for each $n \in \omega$. This observation coupled with the continuity of f guarantees that

$$f(C^*(\mathbb{F}_\infty)) = f(\mathrm{cl}(\{a_n\colon n \in \omega)) \subseteq \mathrm{cl}(f(\{a_n\colon n \in \omega\})) \subseteq \mathrm{cl}\, C_\alpha = C_\alpha.$$

Let $Z = C_\alpha$ and p denote the unital $*$-homomorphism f considered as the homomorphism of $C^*(\mathbb{F}_\infty)$ into Z. Note that $f = i \circ p$, where $i\colon Z = C_\alpha \hookrightarrow \prod\{X_t\colon t \in T\}$ stands for the inclusion.

By construction, $RR(Z) = 0$. Let us show that $p\colon C^*(\mathbb{F}_\infty) \to Z$ is $\mathcal{RR}_0$-invertible in the sense of Introduction. In our situation for any unital $*$-homomorphism $g\colon C^*(\mathbb{F}_\infty) \to X$, where X is a separable unital C^*-algebra of real rank zero, we need to establish the existence of a unital $*$-homomorphism

$h\colon Z \to X$ such that $g = h \circ p$. Indeed, by definition of the set $\mathcal{A}$, we conclude that $g = f_t$ for some index $t \in T$ (in particular, $X = X_t$ for the same index $t \in T$). Next observe that $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$. This allows us to define the required unital $*$-homomorphism $h\colon Z \to X$ as the composition $h = \pi_t \circ i$. This completes the proof of $\mathcal{RR}_0$-invertibility of p and as a consequence (see Introduction) of the universality of Z. □

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Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada

E-mail address: chigogid@math.usask.ca